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Stability calculations for piecewise-smooth delay equations

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Abstract

This paper describes a new method for computing the stability of nonsmooth periodic orbits of piecewise-smooth dynamical systems with delay. Stability computations for piecewise-smooth dynamical systems without delay have previously been performed using discontinuity mappings to ‘correct’ the linearized period map. However, this approach is less convenient for systems with delays due to the infinite dimensional nature of the problem. Additional problems arise due to the discontinuity propagation properties of delay differential equations. The method proposed is based around a multi-point boundary value solver, which allows the correct linearized period map to be constructed directly. We present numerical examples showing the rapid convergence of the method and also illustrate its use as part of a numerical bifurcation study.

1 Introduction

In recent years there has been a surge of interest in piecewise-smooth (PWS) dynamical systems which has produced a wealth of PWS models [Doole and Hogan, 1996; Fossas and Olivar, 1996; Matsumoto, 1984; Piiroinen, 2002] and a large body of information on possible bifurcation scenarios [di Bernardo et al., 2007; Kuznetsov et al., 2003; Wiercigroch and de Kraker, 2000; Zhusubaliyev and Mosekilde, 2003]. PWS functions are often used as idealizations or simplifications of smooth nonlinear functions to allow easier mathematical analysis, for example in cell-cycle models [Tyson and Novák, 2002] or neuronal spiking [Coombes and Osbaldestin, 2000]. Alternatively, in other models PWS behavior is significant because it is intrinsic to the system being studied, for example sliding mode control [Edwards and Spurgeon, 1998; Young et al., 1999] or analogue-digital interfaces [Kollár et al., 2001].

For the analysis of smooth dynamical systems there are many powerful numerical continuation tools available (AUTO [Doedel et al., 1998] and MatCont [Dhooge et al., 2006] are notable examples). These tools allow detailed bifurcation studies to be performed, where the dynamics is investigated under variation of system parameters. Until recently, similar software for PWS-ODEs has not been available. However, with the advent of TC-HAT [Thota and Dankowicz, 2008] and SlideCont [Dercole and Kuznetsov, 2005] it is now possible to investigate the dynamics of PWS-ODEs with continuation techniques. These software packages have been used successfully on a range of applications such as a model of a cell-cycle [Thota and Dankowicz, 2008] and a model of a MEMS device with impacts [Kang et al., 2007].

This paper provides the groundwork for the numerical bifurcation analysis of PWS systems to go a step further and consider the effects of delay. The effects of delay are becoming increasingly important in many fields, such as the modeling of biological processes [Breakspear et al., 2006; Colijn and Mackey, 2007] and control engineering [Herrmann, 2001; Pyragas, 1992, 2002; Sieber and Krauskopf, 2008] amongst others. Difficulties arise because delay differential equations (DDEs) are infinite dimensional dynamical systems [Diekmann et al., 1995; Hale and Verduyn Lunel, 1993; Stépán, 1989] (their state is defined on a function space, typically C^1) and so require careful mathematical treatment. Furthermore, the infinite dimensionality means they can exhibit a wide range of complicated dynamics despite seeming innocuous.

As an initial step in performing a numerical bifurcation study of a PWS-DDE, a method for calculating the linear stability (or equivalently the eigenvalues) of a PWS orbit is required. For ODEs, the typical method for doing this is to break the PWS orbit into a number of different intervals where, in each interval, the orbit is smooth (that is, the nonsmooth points of the orbit lie on the boundaries between intervals). The eigenvalues of the linearized flow on each interval are calculated and composed together with a so-called *discontinuity mapping* (or *saltation matrix*) [Nordmark, 1991; Piironen, 2002]. For DDEs, this approach is far from ideal since both the linearized flow map and the discontinuity mapping are now operators on an infinite dimensional space. This leads to discretization problems, particularly if the length of time spent in a specific interval is less than the maximal time delay (in this case the flow map is also non-compact [Diekmann et al., 1995; Hale and Verduyn Lunel, 1993]).

Here, we take the approach of calculating the linear stability of a PWS orbit in a single step by combining the method of Luzyanina and Engelborghs [2002]; Luzyanina et al. [1997, 2001] for stability calculations of smooth DDEs with a multi-point boundary value solver. By taking this approach, we avoid the problems that occur when applying the standard ODE methods directly to DDEs.

In Sec. 2 we provide a precise definition for a PWS-DDE by extending an existing definition for PWS-ODEs and show how periodic orbits can be found as the solution of a related multi-point boundary-value problem (MP-BVP). This MP-BVP is in turn used as the basis for stability computations. Numerical examples of stability computations are given in Sec. 3. In Sec. 4 there is a case-study of a model of regenerative metal cutting with contact losses, where the methods described in this paper are embedded in a numerical continuation setting and used to investigate a grazing bifurcation associated with the onset of micro-chaos. Finally, we summarize this work in Sec. 5.

2 Method for Stability Computations

2.1 Definition of a PWS-DDE

A simple example of a PWS-DDE composed of two smooth vector fields is

$$\dot{x}(t) = \begin{cases} f_1(x(t), x(t-\tau)) & \text{if } h(x(t), x(t-\tau)) \leq 0 \\ f_2(x(t), x(t-\tau)) & \text{if } h(x(t), x(t-\tau)) > 0 \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, and f_1, f_2, h are sufficiently smooth functions. Transitions between the different vector fields occur on the switching surface defined by $h = 0$. The *initial state* of the system (i.e., the minimum information needed to define a forward trajectory) is the function segment $x(t)$ for $-\tau \leq t \leq 0$ [Hale and Verduyn Lunel, 1993]. While simple models can be written in this form, difficulties immediately arise when the PWS-DDE is composed of more than two vector fields with non-trivial boundaries between them (e.g., when it is possible to pass between a vector field and any other vector field). To overcome this limitation, we extend the definition of a hybrid dynamical system as used by Thota and Dankowicz [2008] to encompass DDEs.

We define a PWS-DDE to be a collection of smooth vector fields

$$\dot{x}(t) = f_m(x^t) \quad (2)$$

indexed by a *mode variable* $m \in M$ where $x^t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is the solution segment $x(t+s)$ for $-\tau \leq s \leq 0$ and M is a finite set. (Equation (2) encompasses distributed delays as well as discrete delays; however, we deal here with discrete delays only.) Associated with this is a collection of *events* $e \in E$ where E is a finite set and e consists of a pair $\pi_e = (m_{\text{in}}, m_{\text{out}})$, a smooth *event function* $h_e(x^t) : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$ and a smooth *jump function* $g_e(x^t) : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \rightarrow \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. The event function $h_e = 0$ implicitly defines a *switching manifold* marking the transition point between the (potentially) different vector fields $(f_{m_{\text{in}}}, f_{m_{\text{out}}})$ and the jump function g_e determines the instantaneous change of state that occurs upon impact with the switching manifold. The minimal *state* needed to uniquely identify a particular trajectory of the system starting at time t_0 is thus x^{t_0} along with the mode m at time t_0 .

A periodic orbit of a PWS-DDE is defined as a collection of *solution intervals* $x_i(t)$ and corresponding modes m_i . In each interval, for convenience, time is shifted such that $-\tau \leq t \leq T_i$, where T_i is the length of time spent in that interval. Furthermore, each end point $x_i^{T_i}$ (really an end *function segment*) lies on the switching manifold defined by $h_{e_i} = 0$, and $x_i(t)$ for $0 \leq t \leq T_i$ is a solution of the vector field f_{m_i} . Finally, the individual solution segments are related by $x_{i+1}^0 = g_{e_i}(x_i^{T_i})$ with periodic extension in i .

We introduce the notion of a *signature* of a periodic orbit defined as $\Sigma = \{m_i, e_i\}$, which is an ordered collection of modes and events that the periodic orbit passes through in one period. This allows us to formulate a particular periodic orbit of the PWS-DDE as a multi-point boundary value problem (MP-BVP) that is determined by the signature of the orbit.

2.2 Periodic Orbits as Solution of MP-BVP

Any periodic orbit of a PWS-DDE can be found as the solution to a related multi-point boundary value problem (MP-BVP). The definition of the MP-BVP comes from the signature $\Sigma = \{m_i, e_i\}$ of the orbit in question and takes the form

$$\dot{x}_i(t) = f_{m_i}(x_i^t) \quad \text{for } 0 \leq t < T_i, \quad (3a)$$

where $i = 0, 1, \dots, N-1$, with N scalar boundary conditions given by

$$h_{e_i}(x_i^{T_i}) = 0 \quad (3b)$$

and N function boundary conditions

$$x_{(i+1) \bmod N}^0 = g_{e_i}(x_i^{T_i}). \quad (3c)$$

The final boundary condition of (3c), $x_0^0 = g_{e_N}(x_N^{T_N})$, is effectively a periodicity condition.

Typically, the length of time T_i spent in each interval is not known a priori. Consequently, when (3) is used to calculate a periodic orbit, it is convenient to rescale each vector field so that the time spent in each is unity. Thus, the true time spent in each interval appears as a parameter in the MP-BVP to be determined by the computation:

$$\dot{u}_i(t) = T_i f_{m_i}(u_i^t) \quad \text{for } 0 \leq t < 1, \quad (4a)$$

$$h_{e_i}(u_i^1) = 0, \quad (4b)$$

$$u_{(i+1) \bmod N}^0 = g_{e_i}(u_i^1), \quad (4c)$$

where $u_i(t) = x_i(T_i t)$ and u_i^t is the function segment $u_i(t+s)$ for $-\tau/T_i \leq s \leq 0$.

2.3 Collocation

To solve the MP-BVP defined by (4) for a particular solution, we discretize the equations using the method of collocation with orthogonal polynomials [Doedel et al., 1991; Engelborghs et al., 2001a], as used by many numerical bifurcation analysis packages including AUTO [Doedel et al., 1998], MatCont [Dhooge et al., 2006] and DDE-BIFTOOL [Engelborghs et al., 2001b]. The idea behind collocation is to use a piecewise polynomial to approximate the solution of the MP-BVP by ensuring that the piecewise polynomial satisfies the underlying differential equation at a set of discrete *collocation points*. When the piecewise polynomial is substituted into the MP-BVP and evaluated at the collocation points, a large system of nonlinear algebraic equations is obtained which can then be solved using standard root finding methods (e.g., a Newton iteration). The rate of convergence of this method is determined by the order of the polynomial used and the number of mesh points used. The convergence of collocation for periodic orbits of DDEs was proved in [Engelborghs and Doedel, 2002]. The convergence of the collocation method for (4) is an open question, however, it is likely that the proof in [Engelborghs and Doedel, 2002] will suffice with small modifications.

For each interval of (4) we define the mesh Π_i as a set of $L_\tau + L + 1$ mesh points over the range $[-\tau/T_i, 1]$, given by

$$\Pi_i := [t_0 = -\tau/T_i, \dots, t_{L_\tau} = 0, \dots, t_{L_\tau+L} = 1],$$

where the interior mesh points $\{t_1, \dots, t_{L_\tau-1}, t_{L_\tau+1}, \dots, t_{L_\tau+L-1}\}$ are arbitrary. The collocation solution $v(t)$ on the interval $[-\tau/T_i, 1]$ is a piecewise vector-valued polynomial, defined on the mesh Π_i , of the form

$$v \in \mathcal{C}([-\tau/T_i, 1], \mathbb{R}^n), \quad v|_{[t_j, t_{j+1}]} \in P_D \quad \text{for } j = 0, \dots, L_\tau + L - 1$$

where P_D is the set of all (vector-valued) polynomials with degree at most D .

To fix the solution over a particular interval we require the value of v at $n(D(L_\tau + L) + 1)$ appropriately chosen points in time (assuming continuity of the collocation solution at the mesh points). Thus, we represent the collocation solution v over a particular interval by a set of values $v_{j+(k/D)}$ at the *representation points* $v_{j+(k/D)} := t_j + (k/D)(t_{j+1} - t_j)$, i.e.

$$v(t) = \sum_{k=0}^D v_{j+(k/D)} P_{j,k}(t) \quad \text{for } t \in [t_j, t_{j+1}]$$

where $P_{j,k}$ is the Lagrange interpolating polynomial given by

$$P_{j,k}(t) = \prod_{r=0, r \neq k}^D \frac{t - t_{j+(r/D)}}{t_{j+(k/D)} - t_{j+(r/D)}}.$$

While the collocation solution is continuous for $t \in [-\tau/T_i, 1]$ it may not be continuously differentiable at the mesh points. Note that we use uniformly distributed representations points for convenience only; there may be some benefits to using other distributions, e.g., Chebyshev [Boyd, 2001; Trefethen, 2000], if a high degree polynomial interpolant is desired.

We choose D collocation points per mesh interval $[t_j, t_{j+1}]$ for $t_j \geq 0$ and require that the vector field (4a) is satisfied exactly at these points; this provides nDL collocation equations to fix the $n(D(L_\tau + L) + 1)$ unknowns. Additionally, there are the N unknowns T_i which specify the interval lengths. The remaining equations needed for a fully specified system are provided by the discretizations of the boundary conditions (4b) and (4c).

The collocation procedure results in a large system of algebraic equations that approximates the solution $u_i(t)$ and T_i of (4). These algebraic equations can be solved using a Newton iteration starting from a sufficiently good initial estimate of the solution.

2.4 Stability of a Periodic Orbit

As with smooth dynamical systems, the (linear) stability of a periodic orbit is determined by the eigenvalues of an appropriate linearized period map ϕ^T . However, since perturbations to an orbit will change the length of time spent in each solution interval, ϕ^T must be constructed in a particular way. Typically, for the finite dimensional (ODE) case, a series of linearized mappings ϕ_i^T is constructed, each of which maps the solution from one switching surface to the next. These linearized mappings are then composed together using a series of discontinuity mappings D_i (alternatively known as saltation matrices), which contain information about the local geometry of the switching surface. The resulting map $\phi^T = \phi_1^T \circ D_1 \circ \dots \circ \phi_N^T \circ D_N$ then takes into account the changes in the lengths of time spent in each solution interval [di Bernardo et al., 2007].

The use of discontinuity mappings to construct the period map ϕ^T is less helpful in the infinite dimensional setting of DDEs for two key reasons. The first is that since the discontinuity mappings are no longer equivalent to matrices (since the state-space is now infinite dimensional), the choice of discretization will be key to the construction of ϕ^T . Furthermore, if the discretizations on each of the solution intervals are not the same (that is, they do not have the same spacing, as is typically the case when using adaptive meshing) there is the added need for one or more interpolation steps. The second reason is that the forward evolution operator for a DDE is not a compact operator on intervals

of time shorter than the time delay, although it is eventually compact over a full period of the orbit. Consequently, this may cause problems when only very short periods of time are spent in a particular solution interval, as can be the case in the vicinity of a grazing bifurcation.

To overcome the aforementioned problems, we take the approach of constructing ϕ^T from the periodic orbit in a single step. The key point is that the lengths of time T_i spent in each solution interval must be considered as extra variables in the period map by normalizing the time spent in each interval to unity (cf. (4)), i.e.,

$$\begin{bmatrix} u_i \\ T_i \end{bmatrix} = \phi^T \begin{bmatrix} \tilde{u}_i \\ \tilde{T}_i \end{bmatrix}$$

where u_i is the solution on the i -th interval scaled to be over the time interval $-\tau/T_i \leq t \leq 1$ as per (4), and the tildes ($\tilde{\cdot}$) denote the value of the variable from the previous time period. Thus, the eigenvalues of ϕ^T can be found and the stability of the periodic orbit determined. As with smooth periodic orbits, the orbit is stable if all of the eigenvalues lie within the unit circle in the complex plane.

The period map ϕ^T can be determined directly from the MP-BVP formulation (4) by changing the periodicity condition $u_0^0 = g_{e_N}(u_N^{T_N})$ to $u_0^0 = g_{e_N}(\tilde{u}_N^{\tilde{T}_N})$. This results in an implicit definition for u_i and T_i in terms of \tilde{u}_i and \tilde{T}_i (actually in terms of \tilde{u}_N and \tilde{T}_N only). This implicit definition for ϕ^T can be discretized using collocation as described in the previous section, then linearized with respect to u_i , T_i , \tilde{u}_i and \tilde{T}_i and resulting in an equation of the form

$$A \begin{bmatrix} \tilde{u}_i \\ \tilde{T}_i \end{bmatrix} + B \begin{bmatrix} u_i \\ T_i \end{bmatrix} = 0$$

where A and B are matrices containing the (discretized) derivatives of the modified MP-BVP (4) evaluated over the periodic orbit. This immediately gives the discretized period map as $\phi_{\text{disc}}^T = B^{-1}A$. The eigenvalues of the resulting matrix determine the stability of the nonsmooth periodic orbit.

2.4.1 Implementation Notes

The matrix B is almost identical to the Jacobian of the original collocation equations used in the Newton iteration to find the original periodic orbit. Consequently, the Jacobian can easily be reused for the purpose of the stability calculation. Also, the matrix A is mostly zeros and so the condensation procedure used in the DDE-BIFTOOL software package [Luzyanina and Engelborghs, 2002] can be employed to reduce the computational complexity of the problem. Further trivial optimizations can be used when all the jump functions g_{e_i} are the identity function, which can result in a significant decrease in computational costs.

One important consideration that has been omitted so far is the discontinuity propagation properties of DDEs; that is, a discontinuity in the derivative of the solution caused by hitting the switching surface (a *primary discontinuity*) will be propagated in the next highest derivative time τ later (a *secondary discontinuity*). For example, a solution $x(t)$ with a discontinuity at $t = 0$ will have a discontinuity in its first derivative at $t = \tau$ and a discontinuity at $t = 2\tau$, etc. Fortunately, since the discontinuities that are propagated occur in higher derivatives each time, this does not cause a significant problem. However, this does necessitate the placement of a mesh point on any discontinuities in the first, second or third derivative of the solution to ensure the accurate computation of ϕ^T . Numerical evidence for this is provided below.

3 Results

We apply the method for computing the stability of PWS-DDEs to the four different examples given below. For each of these examples we compute a reference periodic orbit on an equispaced mesh using the method described in Sec. 2.2 with 1000 mesh points (distributed evenly across the solution intervals) and quintic interpolating polynomials.

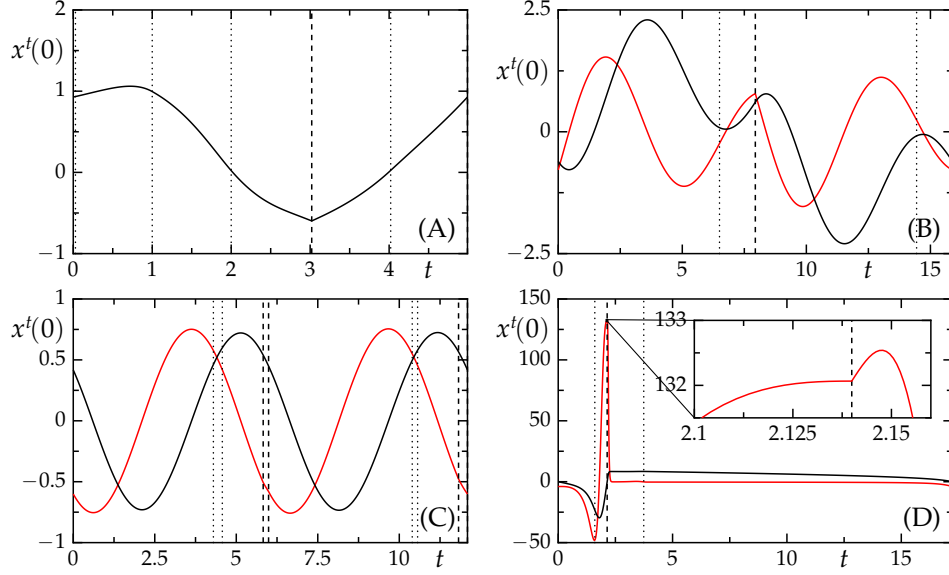


Figure 1: Time series of the reference solutions used for examples (A)–(D). The first derivatives of the solutions are shown in red. The dashed lines denote primary discontinuities (the discontinuities at the switching surface) of the solution and the dotted lines denote the secondary discontinuities (the discontinuities propagated due to the time delay).

To ensure that the stability computations converge, we recompute each reference solution for a range of numbers of mesh points (10 to 500 mesh points) and a range of interpolating polynomial degrees (3rd order to 5th order) to yield a set of approximate numerical solutions. For each approximate solution, its stability is calculated with the method described in Sec. 2.4 and the error in the dominant eigenvalue is determined (against the reference solution).

3.1 First-order DDE with Piecewise-constant Nonlinearity

$$\dot{x}(t) = \alpha x(t) - x^3(t - \tau) + \begin{cases} \beta, & \text{if } x(t - \tau) \geq 0, \\ \gamma, & \text{if } x(t - \tau) < 0. \end{cases} \quad (\text{A})$$

Equation (A) is a simple delayed oscillator that has a discontinuous vector field at the switching surface $x(t - \tau) = 0$. The time series and phase-plane projection of the reference solution used are shown in Fig. 1(A) and Fig. 2(A) respectively. The parameters used are $\alpha = 0.4$, $\beta = -0.15$, $\gamma = 0.68$ and $\tau = 1$.

Figure 3(A) shows the convergence of the dominant eigenvalue for varying orders of interpolating polynomial where two secondary discontinuities are tracked (as indicated by Fig. 1(A)). The method described in Sec. 2.4 converges rapidly to the eigenvalues of the reference solution. As might be expected based on previous studies of collocation methods, the method for computing stability gains an order of accuracy for each increase in the degree of the interpolating polynomial.

However, the rate of convergence of the method depends strongly on the number of secondary discontinuities that are tracked; Fig. 4 shows the convergence of the dominant eigenvalue as the number of discontinuities tracked is changed. The red circles (\circ), blue squares (\square) and black crosses ($+$) each denote different mesh configurations where the one, two and three secondary discontinuities are located at mesh points, respectively. Since the primary discontinuity occurs in the first derivative of the solution, the secondary discontinuities (that are tracked) occur in the second, third and fourth derivatives. Figure 4 shows that mesh points must be placed on the discontinuities in the second and third derivatives to ensure rapid convergence. However, discontinuities in the fourth derivative can be ignored since the convergence is no better than when tracking two secondary discontinuities.

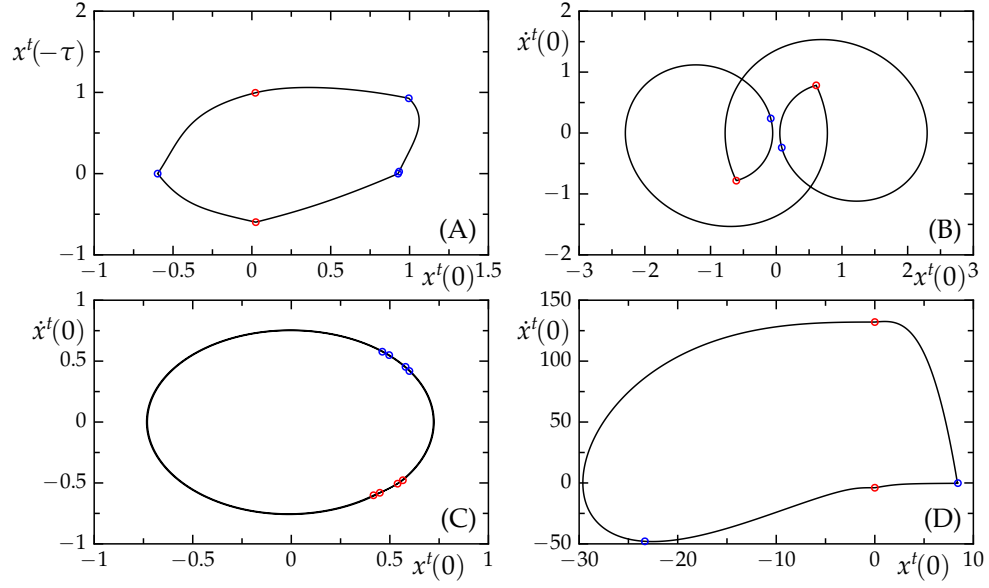


Figure 2: Phase-plane projections of the reference solutions used for equations (A)–(D). The red circles denote the primary discontinuities and the blue circles denote the secondary discontinuities.

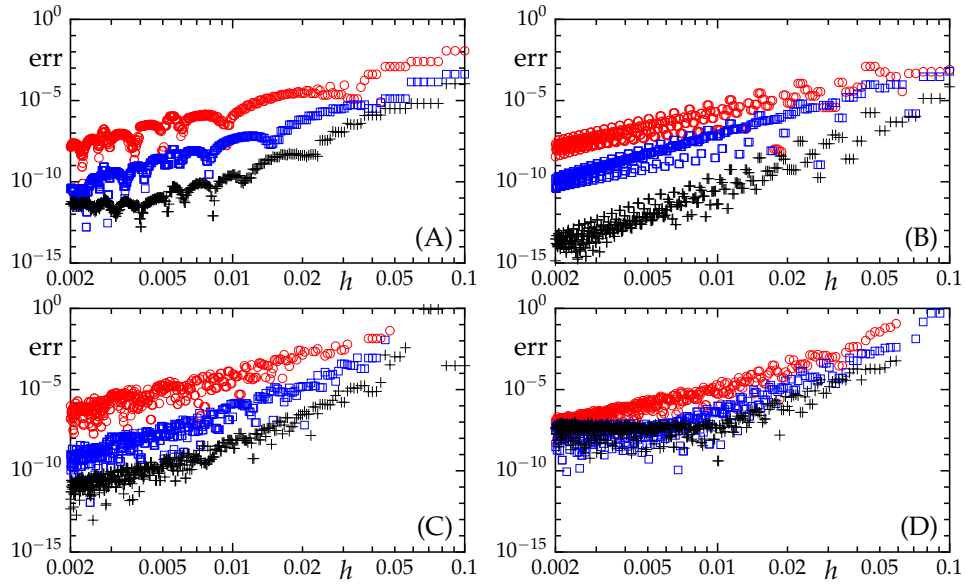


Figure 3: Convergence of the dominant eigenvalues of the reference solutions shown in Fig. 1 and Fig. 2 as the mesh size h tends to zero. The red circles (\circ), blue squares (\square) and black crosses ($+$) mark computations with cubic, quartic and quintic interpolating polynomials respectively.

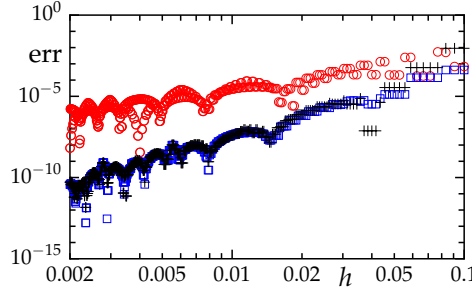


Figure 4: Convergence of the calculated eigenvalues for (A) as the mesh size tends to zero. The red circles (\circ), blue squares (\square) and black crosses ($+$) denote computations where one, two and three secondary discontinuities are tracked, respectively. Hence, all discontinuities in the first, second and third derivatives must be tracked to ensure good convergence.

3.2 Second-order DDE with Piecewise-constant Nonlinearity

$$\ddot{x}(t) + c\dot{x}(t) + x(t) = - \begin{cases} \Theta_1, & \text{if } x(t - \tau) \geq 0, \\ \Theta_2, & \text{if } x(t - \tau) < 0. \end{cases} \quad (\text{B})$$

Equation (B) was studied, with $c = 0$, as a caricature model of the pupil light reflex in [Bayer and an der Heiden, 1998; an der Heiden et al., 1990; an der Heiden and Reichard, 1990]. It was shown to possess a wealth of periodic orbits, both stable and unstable, all connected via a grazing bifurcation [Barton et al., 2006]. Also, a similar system incorporating hysteresis in the switching function was studied in [Colombo et al., 2007].

The reference solution shown in Fig. 1(B) and Fig. 2(B) was computed with the parameter values $c = 0.2$, $\Theta_1 = -\Theta_2 = 1$ and $\tau = 6.5$.

Figure 3(B) shows the convergence of the dominant eigenvalue as the mesh size tends to zero when one secondary discontinuity is tracked. Since the primary discontinuity occurs in the second derivative of the solution, tracking only a single secondary discontinuity (for each primary discontinuity) is justified and this is borne out by the results shown. Again, convergence is rapid and the error is well-bounded until the machine precision is reached.

3.3 Model of Regenerative Metal Cutting

$$\ddot{x}(t) + 2\zeta\dot{x}(t) + x(t) = wf(n - 1 + x(t - n\tau) - x(t)) \quad (\text{C})$$

with a nonsmooth cutting force

$$f(z) = \begin{cases} z + \eta_2 z^2 + \eta_3 z^3, & \text{if } z \geq 0, \\ 0, & \text{if } z < 0. \end{cases}$$

where n is the smallest positive integer such that $n - 1 + x(t - n\tau) - x(t) \geq 0$. Equation (C) has been studied extensively as a model for regenerative metal cutting on a lathe [Kalmár-Nagy et al., 1999, 2001; Stépán, 2001; Stépán and Kalmár-Nagy, 1997]. Previous work has almost exclusively concentrated on the stability of the steady-state solution and comparisons with experimental data. Recently, the periodic “chattering” behavior, where contact between the metal and the cutting tool is repeated lost and regained, has been studied by smoothing the discontinuities [Dombóvári, 2006; Wahi, 2005]. Note that the time delay itself can change discontinuously upon contact loss.

The reference solution shown in Fig. 1(C) and Fig. 2(C) was computed with parameter values $\zeta = 0.04$, $w = 0.09$, $h = 0.00002$ and $\tau = 4.57$. Discontinuities are not apparent in the reference solution since the discontinuities occur in the second derivative of the solution. Convergence results are shown in Fig. 3(C) and are comparable to the results from (A) and (B). The variable time delay is dealt with by

working with the maximum time delay. While this extends the system size, the sparsity of the resulting matrices can be exploited using sparse solvers.

We further study the dynamics of (C) in Sec. 4, where we perform a detailed bifurcation study at the onset of chattering behavior.

3.4 Delayed van der Pol-type Oscillator

$$\ddot{x}(t) + x(t) = \varepsilon g(x(t), \dot{x}(t), x(t - \tau)) \quad (\text{D})$$

where

$$g(x(t), \dot{x}(t), x(t - \tau)) = \begin{cases} (1 - x^2(t))x(t - \tau) + kx(t - \tau), & \text{if } x(t) \geq 0, \\ (1 - x^2(t))\dot{x}(t) + kx(t - \tau), & \text{if } x(t) < 0. \end{cases}$$

Equation (D) is a delayed van der Pol-type oscillator based on the equations studied by Atay [1998]. For the parameters chosen ($\varepsilon = 0.7$, $k = -1$, $\tau = 1.6$), (D) exhibits a bursty time series with small features (as shown in the zoom of Fig. 1(D)), which was specifically chosen to test the limitations of the stability calculations.

Figure 3(D) indeed shows that the performance of the stability computations is significantly worse than the previous three examples. The rapid saturation of the error in the dominant eigenvalue can be explained by a limitation in numerical accuracy (in this particular case, limited to double precision), since some of the terms in the Jacobian of (4) are comparatively large and so swamp small changes in other terms. However, the performance of the algorithm is still acceptable with errors of the order 10^{-7} when the dominant eigenvalue is order one. In an attempt to improve the performance of the stability calculations, adaptive meshing was also implemented for (D). This did not have a significant impact on the order of the errors reported and merely resulted in a tightening of the envelope bounding the errors.

4 Numerical Continuation

The motivation for developing a method for computing the stability of periodic orbits of PWS-DDEs came from the desire to perform numerical bifurcation studies. Thus, we now demonstrate this method within the context of numerical continuation of periodic orbits of (C).

Equation (C) has a trivial steady-state solution $x \equiv 0$, which undergoes a subcritical Hopf bifurcation as w is varied. The smooth periodic orbits that emerge are all unstable. It is known from physical experiments that there is bistability between the stable steady-state solution (corresponding to steady cutting) and a periodic/chaotic state (corresponding the chattering behavior) where contact is repeatedly lost and reestablished with the work piece [Kalmár-Nagy et al., 1999]. Thus, to investigate the extent of the bistable region it is necessary to consider the nonsmooth periodic orbits that correspond to the chattering behavior.

The MP-BVP formulation (4) for computing periodic orbits and the method for calculating the eigenvalues of an orbit have been embedded within a set of numerical continuation routines written in Matlab. The principles of operation have been taken from DDE-BIFTOOL [Engelborghs et al., 2001b] and correspondingly it uses pseudo-arclength parametrization with a secant predictor to trace out branches of orbits in a predictor-corrector manner [Seydel, 1994].

Figure 5 shows the results of a numerical continuation in the parameter w starting at the subcritical Hopf bifurcation at $\|x\| = 0$ with $\zeta = 0.04$, $h = 0.00002$ and $\tau = 4.57$. The branch of smooth unstable periodic orbits exists for decreasing w , until the periodic orbits grow sufficiently and hit the switching surface $1 + x(t) - x(t - \tau) = 0$ (marked as the primary grazing bifurcation in the zoom of Fig. 5). Out of the primary grazing bifurcation we find an unstable periodic orbit with one contact loss per period. Continuation of this orbit reveals that it becomes stable at a period-doubling bifurcation and remains stable as w is increased further.

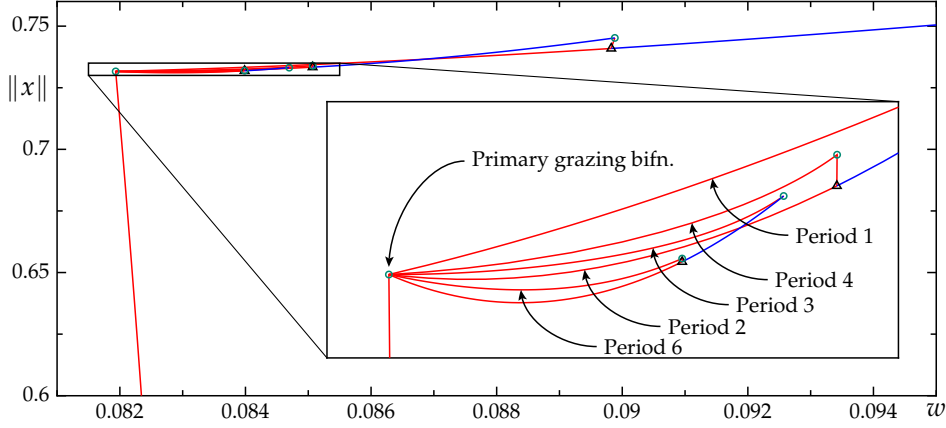


Figure 5: Branches of nonsmooth periodic orbits in (C) for $\zeta = 0.04$, $h = 0.00002$ and $\tau = 4.57$ with varying w . Stable orbits are marked in blue and unstable orbits are marked in red. Grazing bifurcations are marked by green circles (\circ) and period-doubling bifurcations are marked by black triangles (Δ). For parameter values where there is no stable periodic orbit, numerical simulation of (C) shows the existence of a highly localized chaotic attractor.

To switch branches, we find the eigenvector of the period map described in Sec. 2.4 that is associated with the eigenvalue crossing the unit circle at the period-doubling bifurcation. We take two periods of the period-one orbit, perturb it in the direction of the eigenvector and then correct the solution. This results in an orbit on the period-two branch, which has two contact losses per period and can be used to start another continuation run. This period-two branch then encounters a grazing bifurcation, reducing the number of contact losses per period to one and re-stabilizing the branch. The period-two branch remains stable until a second period-doubling bifurcation is encountered. Further continuation of the period-two branch shows that it ends in the primary grazing bifurcation found previously. After a second branch switch, the continuation of the unstable period-four branch (where the orbits have two contact losses per period) reveals a third grazing bifurcation resulting in an unstable period-four orbit with one contact loss per period. This branch undergoes no further bifurcations and it ends in the primary grazing bifurcation along with the period-two branch.

Numerical simulation in the vicinity of the primary grazing bifurcation was performed with the code DDE_SOLVER [Thompson and Shampine, 2004]. This shows the presences of *micro-chaos*, that is, a highly localized chaotic attractor. In the midst of this a stable period-three orbit was found. Numerical continuation of this orbit also reveals an unstable period-six branch that does not undergo any further bifurcations but again ends in the primary grazing bifurcation.

The results of [Dombóvári, 2006] on a smoothed version of (C) also shows the presence of a chaotic attractor in the vicinity of the primary grazing point. However, in the smoothed system the onset of the chaotic region is due to a period-doubling cascade, which we do not find here. Instead we find that the chaotic attractor is the result of the primary grazing bifurcation (which cannot occur in the smoothed system). This shows that although the dynamics of the smoothed and nonsmooth system look similar (i.e., they both contain a highly localized chaotic attractor), the chaotic attractors actually occur via different mechanisms; a point that would have been missed without using nonsmooth continuation.

5 Summary

In this paper we have presented a new method for computing the stability of periodic orbits of piecewise-smooth delay differential equations (PWS-DDEs). The stability of an orbit is computed in a single step, in contrast to the existing methods for ODEs where the mappings from switching surface to switching surface are constructed individually. Performing this computation in a single step has the advantage of alleviating discretization errors and problems associated with non-compactness of the evolution operator on short solution intervals.

The method was tested on four example problems and rapid convergence of the method was observed in all cases, although a sufficient number of secondary discontinuities must be tracked to ensure convergence. Furthermore, this method was embedded into a numerical continuation procedure, thus enabling detailed bifurcation studies to be performed — something which has not been possible with PWS-DDEs until now. In particular, we highlighted the differences between continuation of the true nonsmooth problem and continuation of a related smoothed problem.

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